Dicke-Hepp-Lieb Superradiant Phase Transition and Independent Modes Model in Quantum Optomechanics

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We show that in the strong coupling linear regime, the dispersive opto-mechanical Hamiltonian system can undergo a Dicke-Hepp-Lieb superradiant phase transition with two critical points. The higher critical point is the usual second order Dicke superradiant phase transition while the lower critical point is a new effect showing similarity to the usual Dicke superradiance. We also show that the mechanical mode near its quantum ground state completely decouples from the optical mode and the optical mode shows similarity to the Bogoluibov like excitations.

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I. INTRODUCTION

Classical optomechanics is an already well established branch of optical engineering. Extensive research in this area has lead to development of micro-electromechanical systems (MEMS) which is an essential component of many high technology devices, ranging from iphones to sensors.

Quantum optomechanics on the other hand aims to achieve complete control of the quantum mechanical interaction between electromagnetic radiation and micro- and nano-mechanical resonators. The three essential elements of technology related to quantum optomechanics are state preparation, coherent control and quantum measurement [1]. This field has developed rapidly since the pioneering theoretical work of Braginsky [2] and Caves [3]. In recent years, a wide variety of quantum optomechanical systems have been studied both theoretically and experimentally, particularly nanomechanical cantilever [4–9], vibrating micro-toroids [10, 11], membranes [12] and ultracold atoms [13–25].These achievements have unlocked the door towards quantum regime of optomechanics with coherent control over its properties. This work is a result of motivation derived from these exciting developments in this area. We show here that linearized form of the dispersive optomechanical Hamiltonian exibits the Dicke-Hepp-Lieb [26–28] type superradiant phase transition. We further show that when the mechanical mode is near its quantum ground state, the mechanical mode and the optical mode can be completely decoupled by an appropriate transformation leading to an independent modes model.

II. DICKE-HEPP-LIEB SUPERRADIANT PHASE TRANSITION MODEL

We consider an optomechanical system consisting of a Fabry-Perot cavity in which one of the mirrors is movable. The motion of the mechanical oscillator (movable mirror) will change the length of the optical cavity and as a result will change the resonance frequency of the cavity. The coupling between the cavity field and the mirror is calculated from the dependence of the cavity resonance frequency on the displacement of the mechanical oscillator.

We start from the rotating frame optomechanical Hamiltonian [1] with dispersive coupling.

$$H = -\delta_c c^\dagger c + \omega_m a_m^\dagger a_m + g_m c^\dagger c (a_m^\dagger + a_m), \qquad (1)$$

where c and a_m are the annihilation operators for the cavity mode and the mechanical mode respectively. $\delta_c = \omega_p - \omega_c$ is the detuning of the cavity resonance frequency (ω_c) from the pump frequency (ω_p). Also ω_m is the frequency of the mechanical oscillator and $g_m = \epsilon\omega_m$, $\epsilon = \left(\frac{x_0}{\omega_m}\right)\frac{d\omega_c}{dx}|_{x=0}$ is the dimensionless coupling parameter. Here $x_0 = \sqrt{\frac{\hbar}{2m_{eff}\omega_m}}$ is the zero point motion of the mechanical mode and m_{eff} is the effective mass of the mechanical oscillator. The parameter g_m is the optomechanical single photon coupling strength. The last term of the Hamiltonian (Eqn.1) is parametric coupling where a mechanical displacement controls the frequency of the cavity resonance.

Also $\frac{d\omega_c}{dx}|_{x=0} = -\frac{\omega_c}{L}|_{x=0}$ with L as the length of the cavity. The coupling between the cavity field and the mechanical oscillator is of dispersive type. The Hamiltonian of Eqn.1 is based on some approximations. In order to minimize the quantum fluctuations associated with the various modes of the vibrating mirror, we consider the mirror as a single quantum-mechanical harmonic oscillator with frequency ω_m and mass m_{eff}. Experimentally this approximation can be realized, if we use a bandpass filter in the detection loop, so that the frequencies are limited to a narrow bandwidth which includes a single mechanical resonance. The optical cavity is usually taken to be of high quality factor. The optomechanical coupling is a result of the radiation pressure exerted by the cavity field on the movable mirror. We also assume that the optical cavity field consists of only a single mode. This approximation is valid when $\omega_m << c/2L$. Here we are interested in studying the fluctuation dynamics. Most of the experiments to date can be described using the regime of "linearized optomechanics". To this end, we work in a displaced picture, $a_m \rightarrow a_{ms} + a_m$

and $c \to c_s + c$, where a_{ms} and c_s are the large steady state values of the mechanical mode and the cavity mode respectively.

Without loss of generality, we take the steady state amplitudes a_{ms} and c_s as real. Retaining only the terms quadratic in the fluctuations a_m and c , the Hamiltonian is rewritten as,

$$\begin{aligned} H &= (-\delta_c + 2a_{ms}g_m)c^\dagger c + \omega_m a_m^\dagger a_m \\ &+ g_m c_s (c + c^\dagger)(a_m^\dagger + a_m). \end{aligned} \tag{2}$$

The Hamiltonian now describes the coupling between the fluctuations in the mirror mode and that in the cavity mode. The term $g_m c_s$ is a laser tunable effective coupling connecting the mechanical oscillator and the cavity field. Retaining only quadratic terms in the Hamiltonian amounts to linearizing the equations of motion for a_m and c. The Hamiltonian of Eqn.2 is similar to the single mode Dicke type Hamiltonian [27]. We now demonstrate that the Hamiltonian of Eqn.2 can exhibit Dicke-Hepp-Lieb type superradiant phase transition for two critical points. The term $2a_{ms}g_m c^\dagger c$ is usually neglected and we shown that this term actually gives rise to the second critical point. To this end, we diagonalize the Hamiltonian by introducing the following position and momentum operators.

$$x = \frac{1}{\sqrt{2\omega_m}}(a_m^\dagger + a_m), p_x = i\sqrt{\frac{\omega_m}{2}}(a_m^\dagger - a_m) \tag{3}$$

$$y = \frac{1}{\sqrt{2\omega_0}}(c^\dagger + c), p_y = i\sqrt{\frac{\omega_0}{2}}(c^\dagger - c), \tag{4}$$

where, $\omega_0 = -\delta_c + 2a_{ms}g_m$.

This leads to the following Hamiltonian ignoring constant terms,

$$H = \frac{1}{2}\left(\omega_m^2 x^2 + p_x^2 + \omega_0^2 y^2 + p_y^2 + 4\lambda xy\sqrt{\omega_m\omega_0}\right), \tag{5}$$

where $\lambda = g_m c_s$. The Hamiltonian of Eqn.5 can be diagonalized by the following rotation [27]: $x = q_1 \cos(\gamma_1) + q_2 \sin(\gamma_1)$, $y = -q_1 \sin(\gamma_1) + q_2 \cos(\gamma_1)$, where $\tan(2\gamma_1) = \dfrac{4\lambda\sqrt{\omega_m\omega_0}}{\omega_0^2 - \omega_m^2}$, together with the following bosonic modes,

$$q_1 = \frac{1}{\sqrt{2\epsilon_-}}(c_1^\dagger + c_1), p_x = p_1 = i\sqrt{\frac{\epsilon_-}{2}}(c_1^\dagger - c_1), \tag{6}$$

$$q_2 = \frac{1}{\sqrt{2\epsilon_+}}(c_2^\dagger + c_2), p_x = p_2 = i\sqrt{\frac{\epsilon_+}{2}}(c_2^\dagger - c_2). \tag{7}$$

The final diagonalized Hamiltonian ignoring constant terms is,

$$H = \epsilon_- c_1^\dagger c_1 + \epsilon_+ c_2^\dagger c_2, \tag{8}$$

where, $\epsilon_\pm^2 = \frac{1}{2}\left(\omega_m^2 + \omega_0^2 \pm \sqrt{(\omega_0^2 - \omega_m^2)^2 + 16\lambda^2\omega_m\omega_0}\right)$ are the excitation energies of the two independent oscillators (i.e the optical and mechanical modes) illustrating normal mode splitting (NMS). The excitation energy ϵ_- is real only when $\omega_m^2 + \omega_0^2 \geq \sqrt{(\omega_0^2 - \omega_m^2)^2 + 16\lambda^2\omega_m\omega_0}$, or equivalently $g_m \leq \frac{1}{2}\sqrt{\omega_m(-\delta_c + 2a_{ms}g_m)}$. This leads to two critical values of the coupling constant,

$$g_m^\pm = \frac{\omega_m a_{ms}}{4c_s^2} \pm \sqrt{\frac{\omega_m^2 a_{ms}^2}{4c_s^2} - \frac{\omega_m \delta_c}{c_s^2}}. \tag{9}$$

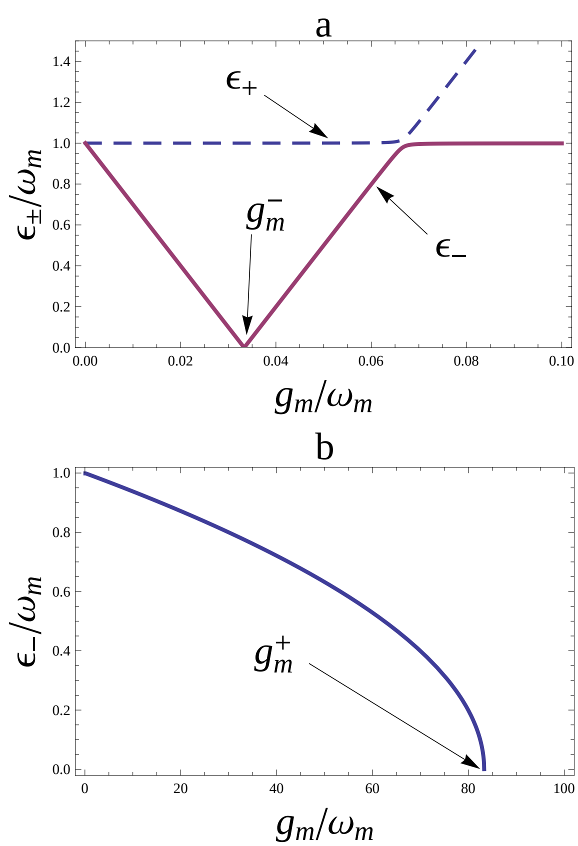

Figure 1: (a): Dimensionless excitation spectra ($\epsilon_\pm$) as a function of dimensionless coupling parameter g_m/ω_m near the lower critical point g_m^- for $a_{ms} = 15$, $c_s = 0.3$ and $\delta_c = \omega_m$. (b): Dimensionless excitation spectrum ϵ_- as a function of dimensionless coupling parameter g_m/ω_m near the higher critical point g_m^+ for the same parameters as in Fig.1(a).

In typical experimental situations, $\delta_c \approx \omega_m$ and $c_s << a_{ms}$. Consequently the two critical points are given as, $g_m^- \approx \frac{\delta_c}{8a_{ms}}$ (first critical point) and $g_m^+ \approx \frac{\omega_m a_{ms}}{2c_s^2} - \frac{\delta_c}{8a_{ms}}$ (second critical point). Fig. 1(a) shows the excitation spectra near the first critical point. For $\omega_0 > \omega_m$, we

identify the two branches as "phononic" (ϵ_-) and "photonic" (ϵ_+) according to the nature of the excitation at zero coupling. From the figure, we see that as the coupling approaches the first critical point g_m^-, the excitation energy of the phononic mode vanishes ($\epsilon_- \to 0$ as $g \to g_m^-$). On increasing the coupling strength further i.e $g_m > g_m^-$, $\epsilon_- \to \omega_m$ (superradiance in the phononic branch). In contrast the photonic branch ϵ_+ initially remains almost constant at ω_0 and then increases to high values for $g_m > g_m^-$. Vanishing of the ϵ_- branch at $g_m = g_m^-$ implies cooling. Fig.1 (b) displays the ϵ_- branch near the second critical point. Vanishing of ϵ_- at g_m^+ is the usual second order Dicke-Hepp-Lieb type quantum phase transition. The essential difference between the lower critical point and the upper critical point is that for g_m^-, the Hamiltonian of Eqn. 2 describes both the normal (left of g_m^-) and the superradiant (right of g_m^-) phase while the Hamiltonian of Eqn. 2 is no longer valid for $g_m > g_m^+$. If $\delta_c < 0$, the first critical point vanishes. For $\omega_0 < \omega_m$, the ϵ_- branch becomes the photonic excitation while ϵ_+ becomes the phononic excitation. This leads to a superradiance in the photonic branch.

III. INDEPENDENT MODES MODEL

We have seen in the previous section that the mechanical and the optical mode hybridizes and the physics is described by two coupled harmonic oscillators. Essentially, the system comprises of coupled fluctuations sitting on top of large steady state values. The single photon strong coupling ($g_0/\omega_m \geq 1$) and many photon weak coupling ($g_0/\omega_m < 1$) has been studied previously [29, 30]. However, many photon, single phonon (harmonic oscillator near quantum ground state) and weak coupling has not been studied earlier. Now in case the mechanical oscillator is near its quantum ground with very small number of phonons as in recent experimental breakthrough [31] , we are no longer able to use the linearized picture $a_m \to a_{ms} + a_m$. The quantum ground state of the mechanical oscillator can be reached for large ω_m [1]. Typically for achieving the quantum ground state, the ratio $g_m/\omega_m \approx 0.24$ [31]. Interestingly, in the limit of $\omega_m >> g_m, \delta_c$ and $a_{ms} = 0$, the eigenvalues $\epsilon_+ \approx \sqrt{\delta_c(\delta_c + \frac{4g_m^2 c_s^2}{\omega_m})}$ and $\epsilon_- \approx \omega_m$, i.e the coupling shifts the cavity resonance while the mechanical oscillator is unaffected. We then need to describe a system in which we have a strong cavity field coupled to a mechanical mode near its quantum ground state with high mechanical frequency. We start from the Hamiltonian of Eqn. 1 and decouple the optical and mechanical degrees of freedom by making a transformation of the type:

$$\tilde{H} = e^S H e^{-S}, \tag{10}$$

where,

$$S = c^\dagger c \frac{g_m}{\omega_m}(a_m^\dagger - a_m). \tag{11}$$

In condensed matter physics, this is called polariton transformation [32]. This leads to the following decoupled Hamiltonian,

$$\tilde{H} = -\delta_c c^\dagger c + \omega_m a_m^\dagger a_m - \frac{g_m^2}{\omega_m} c^\dagger c c^\dagger c. \tag{12}$$

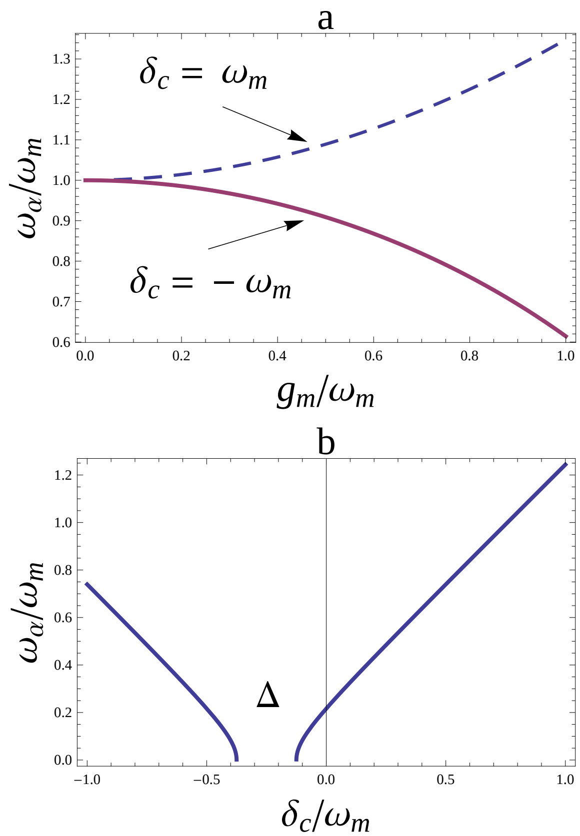

Figure 2: (a): Dimensionless optical excitation spectrum ω_α as a function of g_m/ω_m for $c_s = 0.3$ and two values of the detuning $\delta_c = \pm\omega_m$. (b): Dimensionless optical excitation spectrum ω_α as a function of δ_c/ω_m for $g_m c_s = 0.25\omega_m$. Δ is the gap which can be coherently controlled by the coupling strength and the steady state laser field.

The Hamiltonian of Eqn.12 is now decoupled and the mechanical mode is like a harmonic oscillator while the optical mode behaves like an anharmonic oscillator. Note that the nonlinearity in the optical mode is like kerr nonlinearity. This Hamiltonian has been shown to produce photon blockade [29]. In our working regime, $g_0/\omega_m << 1$ and hence photon blockade is negligible. We still have to diagonalize the cavity part of the Hamiltonian. We are now interested in the fluctuations of the optical mode around the mean field. We again work in a displaced picture, $c \to c_s + c$, where c_s is the steady

state value of the cavity mode. As before we retain terms which are no more and no less than quadratic in the fluctuation operators c. The resulting cavity Hamiltonian ignoring the constant terms is,

$$\tilde{H}_c = -\Delta_c c^\dagger c - \frac{c_s^2 g_m^2}{\omega_m}(c^\dagger c^\dagger + cc), \tag{13}$$

where, $\Delta_c = \delta_c + \frac{4c_s^2 g_m^2}{\omega_m}$ is the shifted cavity detuning. We now introduce the transformations $c = v\alpha^\dagger - u\alpha$, $c^\dagger = v\alpha - u\alpha^\dagger$ in terms of coefficients u, v and the new bosonic operators α and $\alpha^\dagger$. The new operators satisfy $[\alpha, \alpha^\dagger] = 1$ and the coefficients u and v satisfy the constraint $u^2 - v^2 = 1$. We take the coefficients u and v as real. After substituting the transformations in Eqn.(13) and requiring that the coefficients of the off-diagonal terms has to vanish leads us to the final diagonalized cavity Hamiltonian.

$$\tilde{H}_c = \omega_\alpha \alpha^\dagger \alpha, \tag{14}$$

where, $\omega_\alpha = \sqrt{\delta_c^2 + \frac{12 g_m^4 c_s^4}{\omega_m^2} + \frac{8 g_m^2 c_s^2 \delta_c}{\omega_m}}$. Interestingly for $g_m << \omega_m$, $\omega_\alpha = \sqrt{\delta_c\left(\delta_c + \frac{8 g_m^2 c_s^2}{\omega_m}\right)}$, similar to Bogoluibov like excitations [33]. The optical excitation spectrum ω_α is shown in Fig.2(a) as a function of g_m/ω_m for two values of the detuning $\delta_c = \pm\omega_m$. The optical spectrum as a function of δ_c/ω_m is shown in Fig.2(b). The optical gap $\Delta = 4 g_m c_s/\omega_m \sqrt{|\delta_c|/\omega_m}$ is conveniently controlled by the coupling strength g_m. The model proposed here to observe the new opto-mechanical effects are already experimentally feasible [31].

IV. CONCLUSIONS

In conclusion, we have demonstrated that a dispersively coupled opto-mechanical system here is capable of producing phonon or photon superradiance similar to the Dicke-Hepp-Leib type for two critical coupling constants. In addition to the usual Dicke type superradiance, we have shown a new type of superradiance effect is also possible in this system for very low value of the coupling between the optical and mechanical mode. We also show that the mechanical degree of freedom and the optical degree of freedom can be completely decoupled when the mechanical mode is near its quantum ground state. The decoupled optical mode shows similarity to the usual Bogoluibov like modes known in Bose-Einstein condensates.

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